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Wave Optics of the Spherical Gravitational Lens Part I: Diffraction of a Plane Electromagnetic Wave by a Large Star

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Abstract

This paper gives the Poynting vector of a plane electromagnetic wave diffracted by the gravitational field of a large spherical body (large compared to its Schwarzschild radius) and shows in detail how this body works as a gravitational lens. The most interesting results are (1) an extreme amplification of intensity near to the axis of symmetry in the far field behind the body, with a factor of 10 times the Schwarzschild radius divided by the wavelength of the light, and (2) the appearance of double images, differing in shape and position from the predictions of geometrical optics.

1. *Introduction*

The gravitational lens effect of a spherical star (galaxy, . . .) has been investigated by several authors (Refsdal, 1965; Liebes, 1964; Bourassa et al., 1973), but only in the framework of geometrical optics. The disadvantage of all these papers is the fact that all interesting things happen in the region of interference of two rays, where geometrical optics fails to be valid or at least some of its predictions rest on a wavering foundation.

In this paper we shall deal with one important part of the wave-theoretical treatment of the gravitational lens, namely, the scattering of a plane electromagnetic wave by a spherical star. In the language of geometrical optics, we shall consider the image of a very distant star. The Schwarzschild radius of the gravitational lens is assumed to be small compared to its radius but large compared to the (flat space) wavelength of the incident wave.

We start with a short account of notation and Maxwell's equations for the particular symmetry of a plane wave in a spherical background metric. The formulas are given without proof; the details can be found in our first paper (Herlt & Stephani, 1975). We then, in Sec. 3, derive the important formula (3.9) for the diffraction field, which is the starting point for all further considerations. Those interested only in the main results should look at Fig. 3. Note that outside the region of interference geometrical optics holds and inside this region (7.7) is true, and should start reading at Sec. 8.

2. *The General Solution of Maxwell's Equations with the Symmetry of a Plane Wave*

In the background of the Schwarzschild metric

$$\begin{gathered} ds^2 = r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2) + [(r-1)/r](dv^2 - dt^2) \\ v = r + \ln(r-1) \end{gathered} \tag{2.1}$$

the general solution of Maxwell's equations, which corresponds to a monochromatic wave with the symmetry of a plane wave ($E_x = H_y$, all other field components being zero), can be given in terms of a single function P:

$$P(r, \vartheta) = \sum_{n=1}^{\infty} D_n R_n(r) P_n^1(\cos\vartheta) \tag{2.2}$$

The D_n are arbitrary constants, the P_n^1 are the Legendre functions, and the $R_n(r)$ are solutions of the radial equation

$$\frac{d^2R_n}{dv^2} + \left[\omega^2 - \frac{n(n+1)(r-1)}{r^3}\right] R_n = 0 \tag{2.3}$$

To get the components of the electromagnetic field, one simply has to construct

$$\begin{aligned} \alpha &= -\sin\vartheta \frac{\partial}{\partial\vartheta} \frac{1}{\sin\vartheta} \frac{\partial}{\partial\vartheta}(\sin\vartheta\, P) \\ \beta &= \sin\vartheta \frac{\partial}{\partial\vartheta} \frac{\partial P}{\partial v} + i\omega P \\ \delta &= -i\omega \sin\vartheta \frac{\partial P}{\partial\vartheta} - \frac{\partial P}{\partial v} \end{aligned} \tag{2.4}$$

and to insert the result into

$$\begin{aligned} F_{\vartheta\varphi} &= \alpha \sin\varphi\, e^{-i\omega t}, \quad F_{\vartheta v} = \delta \frac{\cos\varphi}{\sin\vartheta} e^{-i\omega t} \\ F_{\varphi t} &= \delta \sin\varphi\, e^{-i\omega t}, \quad F_{\vartheta t} = \beta \frac{\cos\varphi}{\sin\vartheta} e^{-i\omega t} \\ F_{\varphi v} &= \beta \sin\varphi\, e^{-i\omega t}, \quad F_{vt} = \alpha \frac{r-1}{r^3} \frac{\cos\varphi}{\sin\vartheta} e^{-i\omega t} \end{aligned} \tag{2.5}$$

The coordinates are defined so that distances are measured in units of Schwarzschild radius and so that ω is the ratio of the Schwarzschild radius divided by the (flat space) wavelength.

3. *General Expressions for the Rigorous and Approximative Solutions of the Diffraction Problem*

We now have to adjust the general solution (2.2) to our purposes, i.e., we have to impose boundary conditions. As shown in our preceding paper, the boundary condition at radial infinity that ensures an incoming plane wave gives the amplitudes D_n

$$D_n = \frac{(-1)^n}{2\omega^2} \frac{2n+1}{n(n+1)} e^{i\omega(1/2 - \ln 2)} \tag{3.1}$$

This makes sense only in combination with the specific choice that the radial functions R_n contain an incoming part $R_n^{(\text{in})}$ which for large v is approximately $e^{-i\omega v}$.

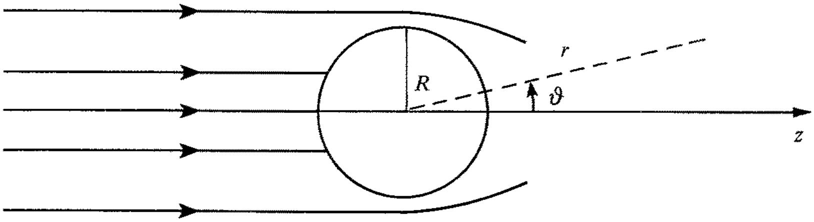

Figure 1—The incident plane wave.

The second boundary condition concerns the field at or near the surface of the star. We demand that the surface completely absorb the incident wave—that is, that no reflection and no coherent reemission take place. It is rather easy to express this condition in terms of the radial functions R_n. The differential equation (2.3) as well as Fig. 2 tells us that an incoming radial wave will

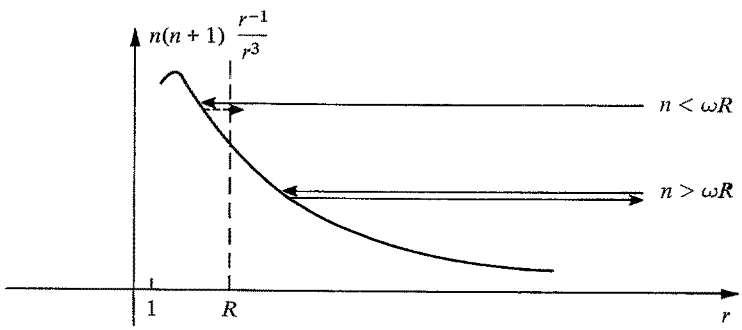

Figure 2—Boundary condition at the surface of the star.

be completely reflected by the "potential" $n(n+1)(r-1)/r^3$ if n is large, and will reach the surface of the star only for $n < N$,

$$N \approx \omega R \tag{3.2}$$

Splitting the radial functions into their ingoing and outgoing parts, we see that for $n > N$ the radial functions R_n are not affected by the star at all; so we need to consider only the R_n of the Black Hole case. For $n < N$ no outgoing wave exists because of the complete absorption, so we have to take the ingoing part of R_n only.

We conclude that

$$P = \sum_{n=1}^{\infty} \frac{(-1)^n}{2\omega^2} \frac{2n+1}{n(n+1)} e^{i\omega(1/2 - \ln 2)} R_n(r) P_n^1(\cos\vartheta)$$
$$- \sum_{n=1}^{\omega R} \frac{(-1)^n}{2\omega^2} \frac{2n+1}{n(n+1)} e^{i\omega(1/2 - \ln 2)} R_n^{\text{out}}(r) P_n^1(\cos\vartheta) \tag{3.3}$$

is the general expression for the exact solution of the diffraction problem.

We now introduce some approximations which simplify this general expression and facilitate computations. Instead of the radial functions $R_n(r)$ we take the functions $\overset{0}{R}_n(r)$ defined by*

$$\overset{0}{R}_n(r) = \frac{2(-i)^{n+1} r^3}{(r^2-1)(r-\frac{1}{2})} e^{i\omega(\ln 2\omega - 1/2)} e^{i\sigma_n} F_n[-\omega, \omega(r-\tfrac{1}{2})]$$
$$\sigma_n = \arg\Gamma(1+n-i\omega) \tag{3.4}$$

where the F_n are solutions of the differential equation

$$\frac{d^2F_n}{dr^2} + \left[\omega^2\left(1 + \frac{2}{r-\frac{1}{2}}\right) - \frac{n(n+1)}{(r-\frac{1}{2})^2}\right] F_n = 0 \tag{3.5}$$

This enables us to express the first part of (3.3) in terms of the confluent hypergeometric function $F[a|c|x]$, the solution of

$$x\frac{d^2F}{dx^2} + (c-x)\frac{dF}{dx} - aF = 0 \tag{3.6}$$

which is regular at $x = 0$. As shown in our previous paper, the equation

$$\overset{0}{P} = \sum_{n=1}^{\infty} D_n \overset{0}{R}_n(r) P_n^1(\cos\vartheta)$$
$$= -(2\pi\omega)^{1/2} \frac{e^{i\pi/4}}{\omega} \frac{r^3}{r^2-1} \frac{1-\cos\vartheta}{\sin\vartheta} e^{i\omega(r+1/2)}$$
$$\{F[1-i\omega|2|-i\omega(r-\tfrac{1}{2})(1-\cos\vartheta)] - F[1-i\omega|2|-2i\omega(r-\tfrac{1}{2})]\} \tag{3.7}$$

holds in consequence of (3.3) and (3.4).

The second approximation concerns the outgoing parts of the radial functions $\overset{0}{R}_n$. Using the well-known properties of F_n (Messiah, 1961) and the

* Some properties of R_n and $\overset{0}{R}_n$ are listed in Appendix A.

Wentzel-Kramers-Brillouin (WKB) method to investigate (3.5), we get for $\omega \gg 1$

$$\overset{0}{R}{}_n^{\mathrm{out}} \approx -(-1)^n e^{-i\omega(1-2\ln 2\omega)} e^{i2\sigma_n} e^{i\omega\tau_n} e^{i\omega[r+\ln(r-1/2)]}$$

$$\begin{aligned}
\sigma_n &= \omega - \omega \ln[\omega(1+a^2)^{1/2}] - a\omega \operatorname{arccot} a \\
\tau_n &= (r^2+2r-a^2)^{1/2} - r + \ln\left[1+\frac{1}{r}+\left(1+\frac{2}{r}-\frac{a^2}{r^2}\right)^{1/2}\right] \\
&\quad - a \arcsin\frac{1-a^2/r}{(1+a^2)^{1/2}} - 1 - \ln 2 + a \arcsin\frac{1}{(1+a^2)^{1/2}} \\
a^2 &= n(n+1)/\omega^2
\end{aligned} \tag{3.8}$$

Putting all the pieces together, we finally obtain

$$\begin{aligned}
P &= \overset{0}{P} - \overset{1}{P} \\
&= -\frac{(2\pi\omega)^{1/2}}{\omega} e^{i\pi/4} \frac{r^3}{r^2-1} \frac{1-\cos\vartheta}{\sin\vartheta} e^{i\omega(r+1/2)} \\
&\times \{F[1-i\omega|2|-i\omega(r-\tfrac{1}{2})(1-\cos\vartheta)] - F[1-i\omega|2|-2i\omega(r-\tfrac{1}{2})]\} \\
&+ \sum_{n=1}^{\omega R} \frac{(2n+1)P_n^1(\cos\vartheta)}{2\omega^2 n(n+1)} e^{i\omega(\ln 2 - 1/2 + 2\ln\omega)} e^{i2\sigma_n} \\
&\times e^{i\omega\tau_n} e^{i\omega[r+\ln(r-1/2)]}
\end{aligned} \tag{3.9}$$

Because this formula is the starting point for all further investigations, we repeat its meaning and make some remarks concerning the physical significance of our approximations. P represents the electromagnetic field of a plane wave diffracted by a star. The radius R of this star should be large ($R \gg 1$), so that the rigorous radial functions R_n can be replaced by the $\overset{0}{R}_n$, and the frequency of the wave should be large ($\omega \gg 1$), so that the WKB method makes sense in the evaluation of the $\overset{0}{R}{}_n^{\mathrm{out}}$. Furthermore (3.8) and (3.9) require $r > R$, so the observer should be outside the star. Generally speaking, the approximation is the better the farther we go away from the star.

4. *Evaluation of the Sum* $\overset{1}{\mathrm{P}}$ *and Interpretation of the Result in Terms of Geometrical Optics*

For the discussion of the electromagnetic field we need more information about

$$\begin{aligned}
\overset{1}{P} &= -\sum_{n=1}^{\omega R} \frac{(2n+1)P_n^1(\cos\vartheta)}{2\omega^2 n(n+1)} e^{i\omega(-1/2+\ln 2+2\ln\omega)} e^{i2\sigma_n} \\
&\times e^{i\omega\tau_n} e^{i\omega[r+\ln(r-1/2)]}
\end{aligned} \tag{4.1}$$

We get this information by substituting for $P_n^1(\cos\vartheta)$ their asymptotic representations

$$P_n^1(\cos\vartheta) = \frac{n}{(2\pi\, n \sin\vartheta)^{1/2}} \{e^{i[(n+1/2)\vartheta - 3\pi/4]} + e^{-i[(n+1/2)\vartheta - 3\pi/4]}\}, \quad n\sin\vartheta \geqslant 1 \tag{4.2}$$

or

$$P_n^1(\cos\vartheta) = \frac{n+\frac{1}{2}}{\cos\vartheta/2} J_1[(2n+1)\sin\vartheta/2], \quad n\sin\vartheta \leqslant 1 \tag{4.3}$$

replacing the sum over n by an integral and evaluating this integral by the method of stationary phase (see Appendix B).

Formulas (4.1) and (4.2) show that the n-dependent part of the phase has the structure

$$S_\pm(n) = \pm\, [(n+\tfrac{1}{2})\vartheta - 3\pi/4] + 2\sigma_n + \omega\tau_n \tag{4.4}$$

The points of stationary phase prove to be

$$\pm \sin\vartheta = \frac{a}{1+a^2}\left[-\frac{a^2}{r} + 1 + \left(1 + \frac{2}{r} - \frac{a^2}{r^2}\right)^{1/2}\right], \quad a^2 = n(n+1)/\omega^2 \tag{4.5}$$

$\overset{1}{P}$ will give contributions only if these points are within the interval $0 \leqslant \vartheta \leqslant \pi$. $1 \leqslant n \leqslant \omega R$. Taking for a the maximum value R, we see that

$$\pm \sin\vartheta = 2/R - R/r \tag{4.6}$$

gives us the boundary of those regions of space which are influenced by $\overset{1}{P}$. This equation admits a simple geometrical and physical interpretation (see Fig. 3). Geometrical optics in the usual linear approximation tells us that rays just grazing the star are given by the equation

$$\begin{aligned} \frac{1}{r} &= \frac{\sin\vartheta}{R} + \frac{(1+\cos\vartheta)^2}{2R^2} \approx \frac{\sin\vartheta}{R} + \frac{2}{R^2}, \quad r \leqslant \frac{R^2}{2} \\ \frac{1}{r} &= -\frac{\sin\vartheta}{R} + \frac{(1+\cos\vartheta)^2}{2R^2} \approx -\frac{\sin\vartheta}{R} + \frac{2}{R^2}, \quad r \geqslant \frac{R^2}{2} \end{aligned} \tag{4.7}$$

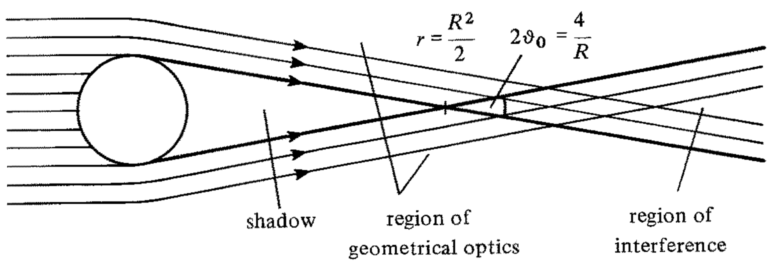

Figure 3—The three different regions of space.

From this we see that the boundary in question coincides with these grazing rays in the forward direction $0 \leqslant \vartheta \leqslant \pi/2$.

A careful analysis of (4.5) shows us that

1. In the shadow (no ray is reaching this region) $\overset{1}{P}$ gives two contributions due to two different points of stationary phase.
2. In the region of geometrical optics (one ray through each point) $\overset{1}{P}$ has one point of stationary phase.
3. In the region of interference (two rays through each point) $\overset{1}{P}$ gives no contributions at all.

This simple result may help to strengthen the reader's confidence in the validity of our approximation procedure.

5. *The Shadow*

People looking for light in the shadow will be disappointed: the two parts of $\overset{0}{P}$–$\overset{1}{P}$ and $\overset{1}{P}$–cancel out: there is no electromagnetic field at all. Because of this rather poor result it does not pay to present the calculations; they run along the same lines as those of Sec. 6.

Believing in wave optics, one would expect a smooth transition between light and shadow and at least a weak field inside the region of shadow. These effects are not covered by our approximations; they need a better asymptotic representation of the functions involved, a refinement of the method of stationary phase, and consideration of terms in ω^{-1}.

We add a remark concerning the physical meaning of $\overset{0}{P}$ and $\overset{1}{P}$, respectively, which will be confirmed later: $\overset{0}{P}$ represents the light which would be found in the absence of the star (only a point singularity or a Black Hole is a cause of diffraction); it consists of two parts corresponding to the two rays crossing each point. $\overset{1}{P}$ represents the light propagating along the very rays that reach the star and are absorbed.

6. *The region of Geometrical Optics*

To simplify calculations, from now on we consider the far field $r \gg 1$ only. One has to be careful with this approximation and should use it only in the final results and not in the intermediate steps of derivation.

By means of the asymptotic representation of the confluent hypergeometric functions,* which is valid for large ω in the region off the axis $\vartheta = 0$, formula (3.9) gives

$$\overset{0}{P} = \frac{e^{i\omega[(r-1/2)(1+\cos\vartheta)/2+1]}}{\omega^2 \sin\vartheta}\left[\frac{r(1-\cos\vartheta)}{r(1-\cos\vartheta)+4}\right]^{1/4}\left(e^{-i\omega N} + ie^{i\omega N}\right)$$

$$-\frac{1}{2\omega^2}\frac{1-\cos\vartheta}{\sin\vartheta}e^{i\omega}\left[e^{-i\omega(r+1/2+\ln 2r)} + ie^{i\omega(r+1/2+\ln 2r)}\right] \qquad (6.1)$$

* Some properties of the confluent hypergeometric functions are compiled in Appendix C.

with

$$N = \tfrac{1}{2}\{(r-\tfrac{1}{2})(1-\cos\vartheta)[(r-\tfrac{1}{2})(1-\cos\vartheta)+4]\}^{1/2} + \ln(1+\tfrac{1}{2}(r-\tfrac{1}{2})(1-\cos\vartheta)+\tfrac{1}{2}\{(r-\tfrac{1}{2})(1-\cos\vartheta)[(r-\tfrac{1}{2}) \times (1-\cos\vartheta)+4]\}^{1/2}) \quad (6.2)$$

The second part of the field, $\overset{1}{P}$, has only one point of stationary phase, which is given by

$$\sin\vartheta = \frac{a}{1+a^2}\left[-\frac{a^2}{r}+1+\left(1+\frac{2}{r}-\frac{a^2}{r^2}\right)^{1/2}\right]$$
$$\omega^{-1} \leqslant a \leqslant R \quad (6.3)$$

Since for large r either $a^2 \gg 1$ or $a^2 \ll r$ holds, (6.3) can be replaced either by

$$a = \cot\tfrac{1}{2}\vartheta, \; a^2 \ll r \quad (6.4)$$

or by

$$2/a - a/r = \vartheta, \; a^2 \gg 1, \; \vartheta \ll 1 \quad (6.5)$$

We will demonstrate the way of reasoning for the first case (6.4). From (4.1), (4.2), and (4.4) it follows that at the point of stationary phase the total phase and its second derivative are

$$S_0 = \omega[-\tfrac{1}{2}+\ln 2+2\ln\omega+r+\ln(r-1)+\tau_n]+2\sigma_n+(n+\tfrac{1}{2})\vartheta-3\pi/4$$
$$= \omega[\tfrac{3}{2}+r+\ln r(1-\cos\vartheta)]-3\pi/4 \quad (6.6)$$

$$\frac{d^2S_0}{dn^2} = \frac{2}{\omega}\sin^2\frac{\vartheta}{2}$$

so that application of (B3) yields

$$\overset{1}{P} = \frac{i}{\omega^2\sin\vartheta}e^{i\omega[r+\ln r(1-\cos\vartheta)+3/2]} \quad (6.7)$$

The approximation $r(1-\cos\vartheta) \gg 4$—which corresponds to (6.4)—simplifies (6.1) to

$$\overset{0}{P} = \frac{1}{\omega^2\sin\vartheta}\left(e^{i\omega[(r-1/2)\cos\vartheta-\ln r(1-\cos\vartheta)]} + ie^{i\omega[r+\ln r(1-\cos\vartheta)+3/2]}\right) - \frac{1}{2\omega^2}\frac{1-\cos\vartheta}{\sin\vartheta}e^{i\omega}\left(e^{-i\omega(r+1/2+\ln 2r)}+ie^{i\omega(r+1/2+\ln 2r)}\right) \quad (6.8)$$

The total field is accordingly given by

$$P = \overset{0}{P} - \overset{1}{P}$$

$$= \frac{1}{\omega^2 \sin\vartheta} e^{i\omega[(r-1/2)\cos\vartheta - \ln r(1-\cos\vartheta)]}$$

$$- \frac{1}{2\omega^2} \frac{1-\cos\vartheta}{\sin\vartheta} e^{i\omega}\left(e^{-i\omega(r+1/2+\ln 2r)} + ie^{i\omega(r+1/2+\ln 2r)}\right) \qquad (6.9)$$

It is rather easy to understand the physical meaning of this result. Owing to (2.4) and (2.5) only the first term will give non-negligible contributions to the electromagnetic field tensor, because in the second term the factor ω^{-2} will not be compensated (no second derivative with respect to r). One need not discuss in detail the electromagnetic field arising from the first term: the surfaces

$$W(r, \vartheta) = (r - \tfrac{1}{2}) \cos\vartheta - \ln r(1 - \cos\vartheta) = \text{const} \qquad (6.10)$$

of constant phase coincide with the surfaces orthogonal to the light rays, and in the short-wavelength approximation $\omega \gg 1$ the Poynting vector is tangential to these rays. So the notation "region of geometrical optics" is justified now.

Surely nobody would have expected a different result. But it may convince the reader once more that in spite of the approximations involved, the results of this section and—what is far more important—of the following sections are reasonable.

7. *The Poynting Vector in the Region of Interference*

In the region of interference $2r \geqslant R^2, R\vartheta \leqslant 2$, the sum $\overset{1}{P}$ gives no contribution and the electromagnetic field can be derived totally from

$$P = -e^{i\pi/4} \frac{(2\pi\omega)^{1/2} r^3}{(r^2-1)\omega} \frac{1-\cos\vartheta}{\sin\vartheta} e^{i\omega(r+1/2)}$$

$$\times \{F[1 - i\omega|2| - i\omega(r - \tfrac{1}{2})(1 - \cos\vartheta)] - F[1 - i\omega|2| - 2i\omega(r - \tfrac{1}{2}]\} \qquad (7.1)$$

It can be shown that the second term is negligible. The electromagnetic field and the Poynting vector due to it are an order of magnitude (factor $\omega^{-1/2}$) smaller than those originating from the first term. Roughly speaking, the second term serves to avoid a singularity of P near the axis $\vartheta = \pi$ and is important only near this axis.

We now are left with the problem of deriving the Poynting vector of the field

$$P = -e^{i\pi/4} \frac{(2\pi\omega)^{1/2} r^3}{\omega(r^2-1)} \frac{1-\cos\vartheta}{\sin\vartheta} e^{i\omega(r+1/2)} F(2) \qquad (7.2)$$

Using the notations

$$F(1) = F[1 - i\omega|1| - i\omega(r - \tfrac{1}{2})(1 - \cos\vartheta)]$$
$$F(2) = F[1 - i\omega|2| - i\omega(r - \tfrac{1}{2})(1 - \cos\vartheta)] \tag{7.3}$$

a straightforward calculation yields (compare Sec. 2) for $r \gg 1$, $\vartheta \ll 1$

$$\alpha = -e^{i(\omega r + \pi/4)}(2\pi\omega)^{1/2} r^2 \vartheta^2 [iF(1) + \omega F(2)]$$
$$\beta = -e^{i(\omega r + \pi/4)}(2\pi\omega)^{1/2}\vartheta \left[\left(ir + \frac{1}{2\omega}\right)F(1) + \left(-\frac{\omega}{2}r\vartheta^2 + \frac{1}{4\omega r} + \frac{i}{2} + \frac{ir\vartheta^2}{4}\right)F(2)\right] \tag{7.4}$$
$$\delta = -e^{i(\omega r + \pi/4)}(2\pi\omega)^{1/2}\vartheta \left\{\left[-ir - \frac{1}{2\omega}\right]F(1) + \left[\frac{i}{2} - ir\frac{\vartheta^2}{4} + \frac{1}{4\omega r}\right]F(2)\right\}$$

For the discussion of the results it is convenient to introduce a coordinate system of cylindrical shape (connected with isotropic coordinates) instead of the Schwarzschild coordinates. In the far field $r \gg 1$ we do this by

$$\bar{r} = r - \tfrac{1}{2}, \quad z = \bar{r}\cos\vartheta, \quad \rho = \bar{r}\sin\vartheta$$
$$ds^2 = \frac{\bar{r}+1}{\bar{r}}\,[d\rho^2 + \rho^2 d\varphi^2 + dz^2] - \frac{\bar{r}}{\bar{r}+1}dt^2 \tag{7.5}$$

In this coordinate system the Poynting vector has the components

$$S_z = -\frac{1}{r^2\vartheta}\,\mathrm{Re}[\delta e^{-i\omega t}]\,\mathrm{Re}\left[\left(\alpha\frac{r-\frac{1}{2}}{r^2} + \frac{\beta}{\vartheta}\right)e^{-i\omega t}\right]$$
$$S_\rho = \frac{1}{r^2\vartheta}\,\mathrm{Re}[\delta e^{-i\omega t}]\,\mathrm{Re}\left[\left(\frac{\alpha}{\vartheta}\frac{r-\frac{1}{2}}{r^2} - \beta\right)e^{-i\omega t}\right] \tag{7.6}$$
$$S_\varphi = 0$$

Time averaging and use of (7.4) gives us the final result

$$\bar{S}_z = \pi\omega F(1)\overset{+}{F}(1) + \pi\omega F(2)\overset{+}{F}(2)\left[\frac{\vartheta^4}{16} - \frac{1}{4r^2}\right]$$
$$-\pi\omega\frac{\vartheta^2}{4}(1 - i\omega)\overset{+}{F}(1)F(2) - \pi\omega\frac{\vartheta^2}{4}(1 + i\omega)F(1)\overset{+}{F}(2)$$

$$\bar{S}_\rho = -\frac{\pi\omega\vartheta}{2r} F(1)\overset{+}{F}(1) + \frac{\pi\omega\vartheta^5}{16} F(2)\overset{+}{F}(2)$$
$$+ \pi\omega\vartheta \left[\frac{i}{2}\omega + \frac{1}{8r^2} + \frac{\vartheta^2}{8}\right] \overset{+}{F}(1)F(2) \tag{7.7}$$
$$+ \pi\omega\vartheta \left[-\omega\frac{i}{2} + \frac{1}{8r^2} + \frac{\vartheta^2}{8}\right] F(1)\overset{+}{F}(2)$$
$$\bar{S}_\varphi = 0$$

Here $\bar{S}_z$, $\bar{S}_\rho$, and $\bar{S}_\varphi$ are the time averaged components of the Poynting vector in the coordinate system (7.5), $F(1)$ and $F(2)$ are the confluent hypergeometric functions defined by (7.3), and the + denotes complex conjugation.

All properties of the diffraction field are hidden in this formula (7.7), and we now have to extract them. This is a little bit complicated, because too many parameters enter into the structure of this interference pattern: the radial distance $r \approx z$, the distance $\rho = r\vartheta$ from the axis $\vartheta = 0$, the frequency ω, and, if we ask for the visual image of the star, the aperture of our telescope. We therefore confine ourselves to the most interesting results. The formulas given below and in the appendices will enable the reader to discuss other details.

One thing should be mentioned first: for our sun the region of interference is outside the planetary system (outside $r = 10^{12}$ km). So all possible observations concern the effects of more distant stars.

8. *Poynting Vector and the Image of the Star*

If we want to discuss the image of the star as seen by a telescope, we have to use physical concepts specific for the telescope and to connect them with the components of the Poynting vector given above.

The intensity dI per unit area

$$dI = \bar{S}_n df^n = \bar{S}_z \rho d\rho d\varphi \tag{8.1}$$

gives us the brightness of the star, and the deflection angle Δ defined by

$$\tan\Delta = -\bar{S}_\rho/\bar{S}_z \tag{8.2}$$

gives us the direction in which the star will be seen. These rather trivial relations are sufficient only if the components $\bar{S}_\rho$ and $\bar{S}_z$ do not change too much along the aperture of the telescope. As we will see later on, the Poynting vector is in fact a rather rapidly oscillating function of the distance ρ from the axis $\vartheta = 0$. It can happen, therefore, that the telescope collects contributions belonging to different deflection angles Δ. In that case the intensity per deflection angle

$$\frac{dI}{d\Delta} = \left|\bar{S}_z \frac{d\rho}{d\Delta}\right| \rho d\varphi = \left|\frac{\bar{S}_z(\bar{S}_z^{2}+\bar{S}_\rho^2)}{\bar{S}'_z\bar{S}_\rho - \bar{S}'_\rho\bar{S}_z}\right| \rho d\varphi,\ \bar{S}'_z = \frac{d\bar{S}_z}{d\rho}, \ldots \tag{8.3}$$

gives us the image of the star.

Formula (8.3) shows that angles Δ with an infinite intensity $dI/d\Delta$ may occur. Owing to the diffraction of the wave by the aperture of the telescope, these infinities will be smoothed out, and at these very angles Δ we will see a peak of intensity.

9. *The Region $r\vartheta^2 \gg 1$: Double Images of Equal Brightness*

Coming from the region of geometrical opticss and crossing the border $\vartheta = 2/R - R/r$ we enter the region of interference at points where $r\vartheta^2 \gg 1$ ($\rho^2 \gg r$) holds, at least for large r. Here we can use the asymptotic representations (C5)-(C8) and simplify (7.7) to

$$\bar{S}_z = \tfrac{1}{2}$$

$$\bar{S}_\rho = -(2/\rho)\sin^2(\omega N - \pi/4) - \rho/4r^2 \tag{9.1}$$

$$N = \rho^2/4r + 1 + \ln(2 + \rho^2/2r)$$

We see that the magnitude of the Poynting vector is 1/2, which is the same as in the region of geometrical optics: the interference does not alter the total brightness of the star.

In the neighborhood of a point r_0, ρ_0, ϑ_0 the component $\bar{S}_\rho$ behaves like

$$\bar{S}_\rho = -\frac{2}{\rho_0}\sin^2\left[\frac{\omega}{2r_0}\rho_0\xi + \text{const}\right] - \frac{\rho_0}{4r_0^2},\ \xi = \rho - \rho_0 \tag{9.2}$$

and the deflection angle

$$\Delta = -\frac{\bar{S}_\rho}{\bar{S}_z} = 2\bar{S}_\rho \tag{9.3}$$

oscillates between $\Delta_1 = \rho_0/2r_0^2 \approx 0$ and $\Delta_2 = 4/\rho_0$ with a period $\delta\rho$ given by

$$\delta\rho = \frac{2\pi r_0}{\rho_0\omega} = \frac{\lambda}{\vartheta_0} \tag{9.4}$$

This period can be small or large compared to the aperture of the telescope.

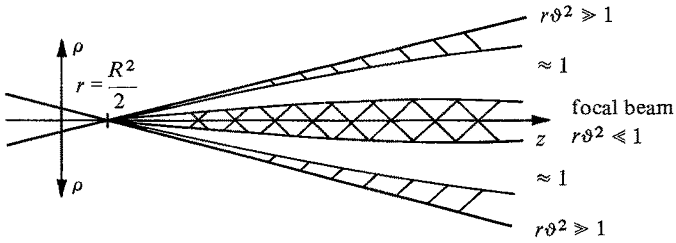

Figure 4—The region of interference and its parts.

1. If the telescope is small, the deflection angle depends on the position of the telescope. This may happen for radio waves scattered by a typical star ($\lambda = 10^{-4}$ km, $r_0 = 10^{16}$ km, $\vartheta_0 = 10^{-6}$, $4/\rho_0 = 4.10^{-10}$ km).

2. If the telescope is large, it will give a distribution of intensity according to

$$\frac{dI}{d\Delta} = \frac{r_0 \rho_0 d\rho d\varphi}{2\omega\rho_0\sqrt{\Delta(4/\rho_0 - \Delta)}} \tag{9.5}$$

compare Fig. 5.

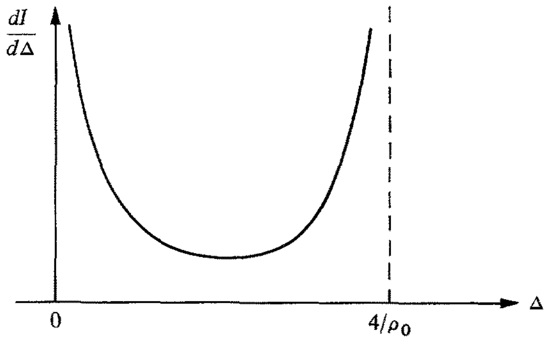

Figure 5—Image of a star as seen by a large telescope.

An observer will see two stars of equal brightness, located at $\Delta = 0$ (no deflection) and at $\Delta = 4/\rho_0$, with a weak bridge in between.

10. *The Region* r$\vartheta^2 \approx$ *1: Double Images of Unequal Brightness*

According to (7.7) and (C5)-(C8) the components of the Poynting vector are

$$\begin{aligned} \bar{S}_z &= \frac{1}{2}\left[1 + \frac{8\cos^2(\omega N - \pi/4)}{r\vartheta^2}\right](1 + 8/r\vartheta^2)^{-1/2} \\ \bar{S}_\rho &= -\frac{2\sin^2(\omega N - \pi/4)}{r\vartheta(1 + 8/r\vartheta^2)^{1/2}} - \frac{\vartheta}{4r}\frac{1 + 8\cos^2(\omega N - \pi/4)/r\vartheta^2}{(1 + 8/r\vartheta^2)^{1/2}} \\ N &= \tfrac{1}{4}r\vartheta^2(1 + 8/r\vartheta^2)^{1/2} + \ln\{1 + \tfrac{1}{4}r\vartheta^2[1 + (1 + 8/r\vartheta^2)^{1/2}]\} \end{aligned} \tag{10.1}$$

Neglecting terms $\rho/2r^2$ we get from this

$$\tan\Delta = \frac{4}{\rho}\frac{\sin^2(\omega N - \pi/4)}{1 + 8\cos^2(\omega N - \pi/4)/r\vartheta^2} \tag{10.2}$$

That shows that in this region the magnitude of the Poynting vector is oscillating between $(1 + 8/r\vartheta^2)^{1/2}/2$ and $(1 + 8/r\vartheta^2)^{-1/2}/2$, while the deflection

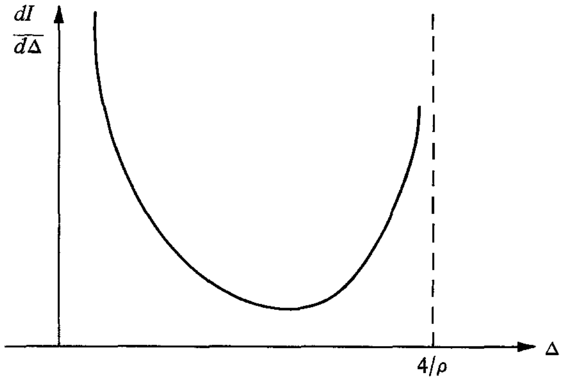

Figure 6—Image of a star as seen by a large telescope, $r\vartheta^2 \approx 1$.

angle Δ oscillates between 0 and $4/\rho$, respectively. Taking into account the variation of ωN, but neglecting variations of ρ elsewhere, the angular distribution of intensity turns out to be

$$\frac{dI}{d\Delta} = \frac{r}{2\omega\rho} \frac{(1+8r/\rho^2)^{1/2}}{(1+2r\Delta/\rho)^2} \frac{\rho d\rho d\varphi}{[\Delta(4/\rho - \Delta)]^{1/2}} \tag{10.3}$$

An observer will see two stars of unequal brightness, located at $\Delta_1 = 0$ and at $\Delta_2 = 4/\rho$, with a ratio of luminosity

$$L_1/L_2 = (1+8r/\rho^2)^2 = (1+8/r\vartheta^2)^2 \tag{10.4}$$

and a weak bridge between both.

11. *The Region* $r\vartheta^2 \ll 1$*: The Focal Beam of Extreme Intensity*

Near the axis of symmetry $\vartheta = 0$ we get from (7.7) the formulas

$$\begin{aligned} \bar{S}_z &= \pi\omega\left[J_0^2(\omega\sqrt{2r}\vartheta) + \frac{r\vartheta^2}{8} J_1^2(\omega\sqrt{2r}\vartheta)\right], \quad r\vartheta^2 \ll 1,\ \omega\sqrt{2r}\vartheta \leqslant 1 \\ \bar{S}_\rho &= -\frac{\pi\omega\vartheta}{2r}\,[J_0^2(\omega\sqrt{2r}\vartheta) + rJ_1^2(\omega\sqrt{2r}\vartheta)] \end{aligned} \tag{11.1}$$

which are valid just at the axis $\vartheta = 0$, and

$$\begin{aligned} \bar{S}_z &= \frac{\sqrt{2}}{\sqrt{r\vartheta}} \cos^2\left(\omega\sqrt{2r}\vartheta - \frac{\pi}{4}\right) + \frac{\sqrt{r\vartheta}}{4\sqrt{2}} \\ \bar{S}_\rho &= -\frac{1}{\sqrt{2r}} \sin^2\left(\omega\sqrt{2r}\vartheta - \frac{\pi}{4}\right) - \frac{1}{r\sqrt{2r}} \cos^2\left(\omega\sqrt{2r}\vartheta - \frac{\pi}{4}\right) \end{aligned} \tag{11.2}$$

where

$$r\vartheta^2 \ll 1, \quad \omega\sqrt{2r}\vartheta \gg 1$$

which are valid off the axis $\vartheta = 0$ and overlap slightly with (11.1).

11.1. *The Light Intensity in the Focal Beam.* The magnitude $|\bar{S}| \approx \bar{S}_z$ of the Poynting vector starts at the axis $\vartheta = 0$ with

$$|\bar{S}| = \pi\omega \quad \text{for } \rho = 0 \tag{11.3}$$

decreases slowly (while oscillating) if ρ increases, becoming

$$|\bar{S}| = \sqrt{\pi\omega} \quad \text{for } \rho = \tilde{\rho} = \sqrt{2r/\pi\omega} = \sqrt{r\lambda/\pi} \tag{11.4}$$

and going further down to its mean value

$$|\bar{S}| = \tfrac{1}{2} \quad \text{for } \rho \gg \sqrt{2r} \tag{11.5}$$

in the region of double images.

Since for a star of about one solar mass the frequency ω can have values between 10^4 (radio waves) and 10^{10} (visible light) or even 10^{14} (γ rays), a gravitational lens can enlarge the brightness of a star for the same remarkable factor. The plane wave is focused into a narrow beam of extreme intensity; the radius of this beam is ρ as given in (11.4). If, owing to a relative motion between source, lens, and earth the earth would be hit by this focal beam, an observer should see a (short) burst of radiation.

The components of the Poynting vector and the intensity are oscillating with a spatial period

$$\delta\rho = \pi\sqrt{r}/\omega\sqrt{2} = \sqrt{r/8}\lambda \tag{11.6}$$

Again it depends on r, λ, and the aperture of the telescope whether these oscillations are observable or not.

11.2. *The Angular Distribution of Intensity.* From (11.2) we get an angular distribution of intensity of the shape

$$\frac{dI}{d\Delta} \approx \frac{(1 + \tan^2 \Delta)\rho d\Delta d\varphi}{2\omega\sqrt{2r}(\tan \Delta + \rho/2r)^2\sqrt{\tan \Delta(4/\rho - \tan \Delta)}} \tag{11.7}$$

where

$$\rho^2 \ll 2r, \rho \gg \sqrt{2r}/2\omega$$

It shows that there are two peaks of intensity, at $\Delta_1 = 0$ and at $\Delta_2 = \arctan 4/\rho$. Near these peaks $dI/d\Delta$ behaves like

$$\begin{aligned} \frac{dI}{d\Delta} &\approx \frac{1}{\sqrt{2}}\left(\frac{r}{\rho}\right)^{3/2} \frac{\rho d\Delta d\varphi}{\omega\sqrt{\Delta}}, \quad \Delta \approx \Delta_1 \approx 0 \\ \frac{dI}{d\Delta} &\approx \frac{\sqrt{\rho^3(\rho^2 + 16)}}{64\omega\sqrt{2r}} \frac{\rho d\Delta d\varphi}{\sqrt{\Delta_2 - \Delta}}, \quad \Delta \approx \Delta_2 \end{aligned} \tag{11.8}$$

That means that the intensity near to the second peak $\Delta = \Delta_2$ is very small; it is smaller the larger Δ_2 becomes. So an observer would essentially see one star (at $\Delta = 0$).

Discussing these formulas one should be aware of the fact that our approximations may be too rough to give the details of the intensity near $\Delta = \Delta_2$. The intensity at this angle is one order of magnitude smaller than that at $\Delta = 0$, and the position Δ_2 of the second peak itself is rather sensitive to small changes in the components of the Poynting vector. So these quantities may change considerably if we take into account terms neglected up to now (terms small compared to $\pi\omega$).

12. *Wave Optics in Comparison with Geometrical Optics*

Geometrical optics, too, predicts a double image of the star, due to rays passin at different sides of the gravitational lens.

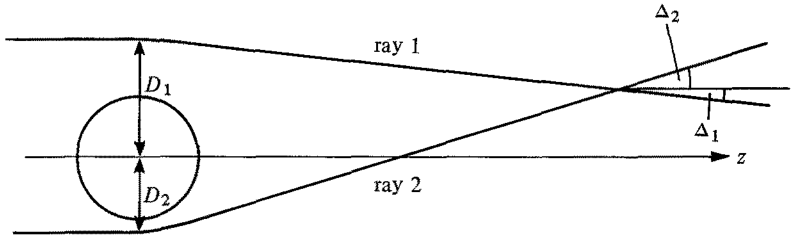

Figure 7—The double image due to geometrical optics.

According to (4.7) the positions of these point images are given by

$$\begin{aligned} \Delta_1 &= \frac{2}{D_1} = \frac{4}{\sqrt{r}} \frac{1}{\sqrt{r\vartheta^2} + \sqrt{r\vartheta^2 + 8}} \\ \Delta_2 &= -\frac{2}{D_2} = -\frac{4}{\sqrt{r}} \frac{1}{\sqrt{r\vartheta^2} - \sqrt{r\vartheta^2 + 8}} \end{aligned} \tag{12.1}$$

and their respective luminosities are (Refsdal, 1965)

$$\begin{aligned} L_1 &= \tfrac{1}{4}[2 + (1 + 8/r\vartheta^2)^{1/2} + (1 + 8/r\vartheta^2)^{-1/2}] \\ L_2 &= \tfrac{1}{4}[-2 + (1 + 8/r\vartheta^2)^{1/2} + (1 + 8/r\vartheta^2)^{-1/2}] \end{aligned} \tag{12.2}$$

Table I helps to compare these positions with those of wave optics, the sum of the luminosities $L_1 + L_2$ with the mean value of $2\bar{S}_z$, and the ratio of L_1/L_2 of the luminosities with the ratio of the values of $dI/d\Delta$ at Δ_1 and Δ_2. Here Δ_1 always denotes the smaller deflection angle, Δ_2 the larger one.

Three points are remarkable:

1. There is a complete agreement concerning the total intensity of the image of the star.

TABLE 1. Predictions of wave optics (W) and geometrical optics (G)

Region		Δ_1	Δ_2	$L_1 + L_2$ or I	L_1/L_2 or $\left(\frac{dI}{d\Delta}\right)_1 \Big/ \left(\frac{dI}{d\Delta}\right)_2$
$r\vartheta^2 \gg 1$	G	$2/r\vartheta$	$-\vartheta$	1	∞
	W	$\rho/2r^2 \approx 0$	$4/r\vartheta$	1	1
$r\vartheta^2 \approx 1$	G	$\frac{4}{\sqrt{r^2\vartheta^2 + 8r} + r\vartheta}$	$\frac{-4}{\sqrt{r^2\vartheta^2 + 8r} - r\vartheta}$	$\frac{1 + 4/r\vartheta^2}{\sqrt{1 + 8/r\vartheta^2}}$	$\frac{\sqrt{1 + 8/r\vartheta^2} + 1 + 4/r\vartheta^2}{-\sqrt{1 + 8/r\vartheta^2} + 1 + 4/r\vartheta^2}$
	W	0	$4/r\vartheta$	$\frac{1 + 4/r\vartheta^2}{\sqrt{1 + 8/r\vartheta^2}}$	$(1 + 8/r\vartheta^2)^2$
$r\vartheta^2 \ll 1$, $\omega\sqrt{2r}\vartheta \gg 1$	G	$\sqrt{\frac{2}{r}}$	$-\sqrt{\frac{2}{r}}$	$\frac{\sqrt{2}}{\vartheta\sqrt{r}}$	1
	W	0	$4/r\vartheta$	$\frac{\sqrt{2}}{\vartheta\sqrt{r}}$	$\frac{64\omega}{r\vartheta^3\sqrt{r^2\vartheta^2 + 16}}$

2. There is a total disagreement concerning the relative intensity of the double images.
3. There is a total disagreement concerning the position of the two images of the star; only for $r\vartheta^2 \gg 1$ the wave optical value of Δ_2 agrees with the Δ_1 of geometrical optics.

Near the axis $\vartheta = 0$ geometrical optics fails to be applicable at all, whereas wave optics gives finite values for all physical quantities.

13. *Concluding Remarks*

The most interesting question is, of course, whether some of the predictions of wave optics of the gravitational lens are observable—not anywhere in the universe, but on our earth.

As the deflection angle $\Delta_2 = 4/\rho$ may have values considerably larger than the usual values occurring in the light deflection by the sun, Δ_2 may have measurable magnitude. But unfortunately the corresponding intensity is smaller the larger Δ_2 becomes.

More promising is the appearance of the focal beam of extreme intensity, as discussed in Sec. 11.1. If the earth happens to pass through this focal beam, a sudden burst of radiation should be seen. The condition to be fulfilled is that the earth approach the axis $\vartheta = 0$ (the line between two stars, one acting as source, the other one acting as lens) up to a distance

$$\tilde{\rho} \approx \sqrt{2r/\pi\omega}$$

compare (11.4) and Fig. 8.

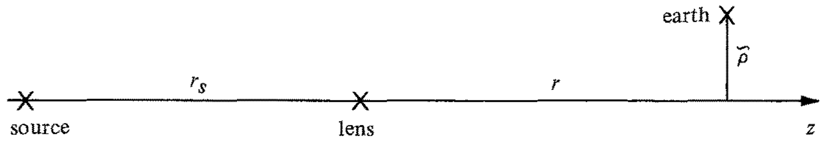

Figure 8—Condition for the earth passing the focal beam.

Moreover, because our calculations were made for an incident plane wave, the distance between source and lens should be large compared to the lens-earth distance ($r_s \gg r$). Refsdal (1965) gave an estimate of the number of passage per year, which is of the order unity—but we need to know first when and where these passages occur to have a chance for observation.

There may be a greater chance of observation for the lens effect by a double star system, one of these stars (the lens) being a neutron star or a black hole. But the plane wave approximation used in this paper fails to be applicable in that case. Our future work will be devoted to these problems.

Appendix A: The Radial Functions R_n and $\overset{0}{R}_n$

The defining differential equations are

$$\frac{d^2R_n}{dr^2} + \frac{1}{r(r-1)}\frac{dR_n}{dr} + \left[\frac{\omega^2 r^2}{(r-1)^2} - \frac{n(n+1)}{r(r-1)}\right] R_n = 0 \qquad \text{(A1)}$$

and

$$\frac{d^2F_n}{dr^2} + \left[\omega^2\left(1 + \frac{2}{r-\frac{1}{2}}\right) - \frac{n(n+1)}{(r-\frac{1}{2})^2}\right] F_n = 0$$

$$\overset{0}{R}_n = 2(-i)^{n+1} r^2 (r-1)^{-2} e^{i\omega(\ln 2\omega - 1/2)} e^{i\sigma_n} F_n \qquad \text{(A2)}$$

$$\sigma_n = \arg\Gamma(1 + n - i\omega)$$

For large r the differential equations for R_n and $\overset{0}{R}_n$ differ only in terms of higher order in r^{-1} (indicated by $\cdots$); the following equation holds for both of them:

$$\frac{d^2R_n}{dr^2} + \left(\frac{1}{r^2} + \ldots\right)\frac{dR_n}{dr} + \left[\omega^2\left(1 + \frac{2}{r} + \cdots\right) - \frac{n(n+1)}{r^2}\left(1 + \frac{1}{r} + \cdots\right)\right] R_n = 0 \qquad \text{(A3)}$$

Calling "ingoing" the part which asymptotically for large r is $e^{-i\omega(r + \ln r)}$ and "outgoing" the part proportional to $e^{i\omega(r + \ln r)}$, one gets

$$R_n^{in} \approx \exp\{-i\omega[r + \ln(r-1) + n(n+1)/2\omega^2 r + \cdots]\}$$

$$\overset{0}{R}{}_n^{in} \approx \exp\{-i\omega[r + \ln(r-1) + n(n+1)/2\omega^2 r + 1/r + \cdots]\} \qquad \text{(A4)}$$

where

$$n \ll \omega r, \; r \gg 1$$

and

$$\begin{aligned} R_n^{\text{out}} &\approx -(-1)^n e^{i\omega T_n} \exp[-2i\omega^2(7/8 + 15\pi/32)/n] \\ \overset{0}{R}{}_n^{\text{out}} &\approx -(-1)^n e^{i\omega T_n} e^{i\omega(1/r - 1/n)} \\ T_n &= r + \ln(r-1) + n(n+1)/2\omega^2 r - 1 + 2\ln 2\omega + \cdots \end{aligned} \tag{A5}$$

All this shows that for large r and n the $\overset{0}{R}_n$ are a good approximation of the R_n.

Appendix B: The Method of Stationary Phase

If the phase $S(n)$ in an integral

$$I = \int A(n) e^{iS(n)} dn, \; n \text{ real} \tag{B1}$$

is a rapidly changing function of n, and $A(n)$ changes only slowly, then the contributions of the integrand will cancel out with the exception of the points n_0 of stationary phase. For these points

$$dS/dn = 0, \quad n = n_0 \tag{B2}$$

holds. The integral can be replaced in a rather good approximation by

$$I = \Sigma A(n_0) e^{iS(n_0)} e^{i\pi/4} \sqrt[+]{\frac{2\pi}{S''(n_0)}} \tag{B3}$$

where S'' denotes the second derivative of S with respect to n and the sum has to be extended over all points of stationary phase lying within the range of integration.

Appendix C: Confluent Hypergeometric Functions

$F[a|c|x]$ is defined as the regular solution of

$$x \frac{d^2F}{dx^2} + (c - x)\frac{dF}{dx} - aF = 0 \tag{C1}$$

It fulfills the following relations:

$$F[a|c|x] = e^x F[c - a|c|-x] \tag{C2}$$

$$\frac{d}{dx} F[a|c|x] = \frac{a}{c} F[a+1|c+1|x]$$
$$= \frac{a}{x} \{F[a+1|c|x] - F[a|c|x]\}$$
$$= \left(\frac{a}{c} - 1\right) F[a|c+1|x] + F[a|c|x]$$
$$= \frac{1-c}{x} \{F[a|c|x] - F[a|c-1|x]\} \tag{C3}$$

The functions used in our paper,

$$\begin{aligned} F(1) &= F[1 - i\omega|1|-i\omega(r - \tfrac{1}{2})(1 - \cos\vartheta)] \\ F(2) &= F[1 - i\omega|2|-i\omega(r - \tfrac{1}{2})(1 - \cos\vartheta)] \end{aligned} \tag{C4}$$

have the asymptotic representations

$$F(1) = \frac{e^{-i(\omega/2)X} e^{i\pi/4}}{(2\pi\omega)^{1/2}} \left(\frac{X}{X+4}\right)^{1/4} \left[\frac{e^{-i\omega N}}{2}\left(1 + \sqrt{\frac{X+4}{X}}\right) - \frac{2ie^{i\omega N}}{\sqrt{X}(\sqrt{X} + \sqrt{X+4})}\right]$$

$$F(2) = -\frac{e^{-i(\omega/2)X} e^{-i\pi/4}}{(2\pi\omega)^{1/2} X} \left(\frac{X}{X+4}\right)^{1/4} (e^{-i\omega N} + ie^{i\omega N}) \tag{C5}$$

where

$$N = \tfrac{1}{2}[X(X+4)]^{1/2} + \ln\{1 + \tfrac{1}{2}X + \tfrac{1}{2}[X(X+4)]^{1/2}\}$$
$$X = (r - \tfrac{1}{2})(1 - \cos\vartheta)$$

valid for $\omega \gg 1$ and $\vartheta > 0$, and the asymptotic representations by means of Bessel functions

$$\begin{aligned} F(1) &= e^{-i\omega X/2}[J_0(2\omega\sqrt{X}) - (i/2)\sqrt{X} J_1(2\omega\sqrt{X})] \\ F(2) &= e^{-i\omega X/2} J_1(2\omega\sqrt{X})/\omega\sqrt{X} \end{aligned} \tag{C6}$$

valid for $\omega \gg 1$ near $\vartheta = 0$ $(X = r\vartheta^2/2)$.

The products of these functions which enter into the components of the Poynting vector are

$$\begin{aligned} F(1)\overset{+}{F}(1) &= \left(\frac{X}{X+4}\right)^{1/2} \frac{1}{2\pi\omega} \left[1 + \frac{4}{X}\cos^2\left(\omega N - \frac{\pi}{4}\right)\right] \\ F(2)\overset{+}{F}(2) &= \left(\frac{X}{X+4}\right)^{1/2} \frac{2}{X^2 \pi\omega^3} \sin^2\left(\omega N - \frac{\pi}{4}\right) \\ \overset{+}{F}(1)F(2) &= i\,\frac{\sin^2(\omega N - \pi/4)}{\pi\omega^2 [X(X+4)]^{1/2}} - \frac{\cos 2\omega N}{2\pi\omega^2 X} \end{aligned} \tag{C7}$$

and

$$
\begin{aligned}
F(1)\overset{+}{F}(1) &= J_0^2(\omega N) + \tfrac{1}{16}N^2 J_1^2(\omega N)\\
F(2)\overset{+}{F}(2) &= (4/\omega^2 N^2)J_1^2(\omega N), \quad N = 2\sqrt{X}\\
\overset{+}{F}(1)F(2) &= (i/2\omega)J_1^2(\omega N) + (2/\omega N)J_0(\omega N)J_1(\omega N)
\end{aligned}
\tag{C8}
$$

Note Added in Proof

While this paper was in press, we became aware of two publications which deal with the (wave theoretical) gain of intensity on the focal line. They are Ohanian, H. C. (1974). *International Journal of Theoretical Physics*, **9**, 425, and Bliokh, P. V. and Minakov, A. A. (1975). *Astrophysics and Space Science,* **34**, L7.

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